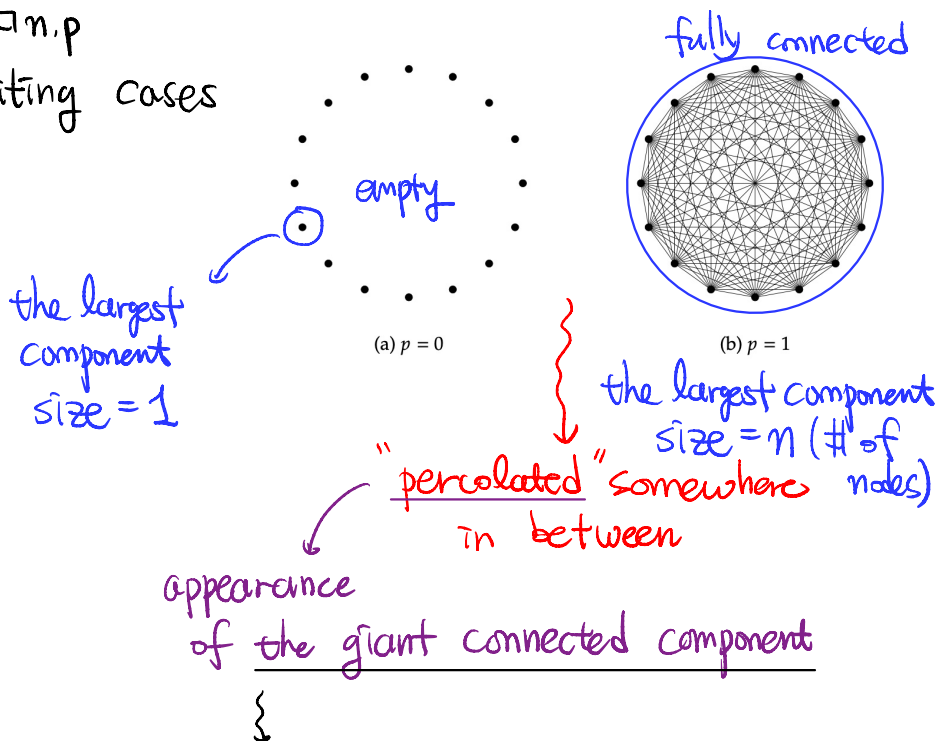


Erdős-Rényi (ER) random graph: Gilbert's version (characterized by the connection probability p)

$\leadsto G_{n,p}$

* limiting cases

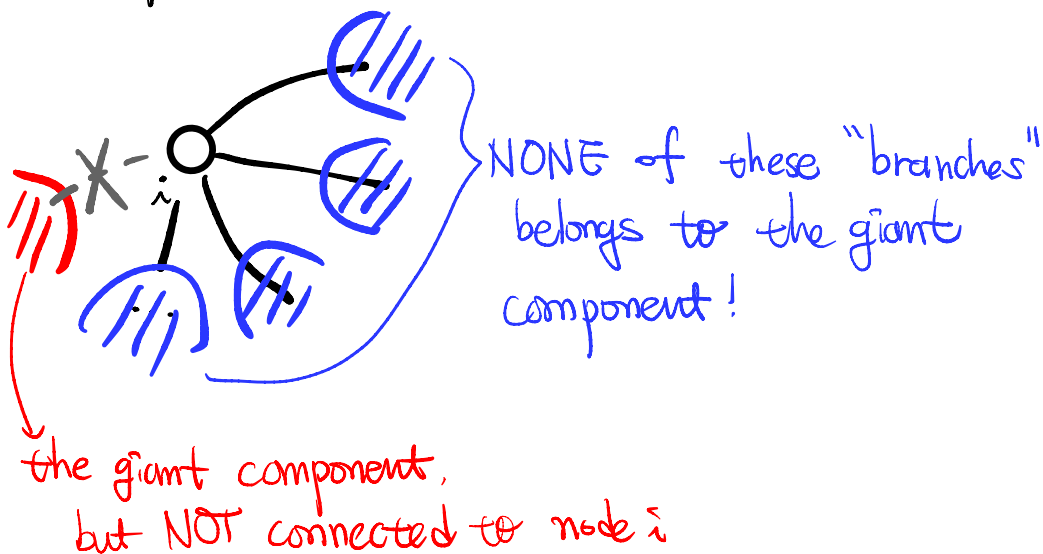


U : the average fraction of nodes in the random graph that do NOT belong to the giant component

(= the probability that a randomly chosen node in the network does not belong to the giant component)

why?
it's more convenient!

For node i not to belong to the giant component, it must not be connected to the giant component via any other node.



So, for each of other node j ,

- i) $\underset{\substack{\uparrow \\ \text{probability} \\ 1-p}}{\bigcirc} \text{---} \text{X} \text{---} \bigcirc_{\substack{\uparrow \\ j}}$: not connected to j , then it's OK, regardless of j 's belonging to the giant component or not!
- ii) $\underset{\substack{\uparrow \\ \text{probability } p}}{\bigcirc} \text{---} \bigcirc_{\substack{\uparrow \\ j}}$: connected to j , then node j must not belong to the giant component!

$$\Rightarrow \boxed{\text{prob}[i) + ii)] = (1-p) + p \times u = 1-p+pu}$$

$\downarrow$
 or
 (either)

$\hookrightarrow$ should hold for all of the other nodes!
 $\downarrow$
 (n-1) nodes

$$\therefore u = (1-p+pu)^{n-1} : \text{a self-consistency equation for } u$$

the average degree $C = p(n-1)$, or $p = \frac{C}{n-1}$

$$\leadsto u = \left[1 - \frac{C}{n-1} (1-u) \right]^{n-1}$$

taking log of both sides,

$$\ln u = (n-1) \ln \left[1 - \frac{C}{n-1} (1-u) \right]$$

$$\nearrow \approx - (n-1) \left[\frac{C}{n-1} (1-u) \right] = -C(1-u)$$

for $\frac{C}{n-1} \ll 1$, $\ln(1-x) \approx -x$
 (large n or sparse network limit)

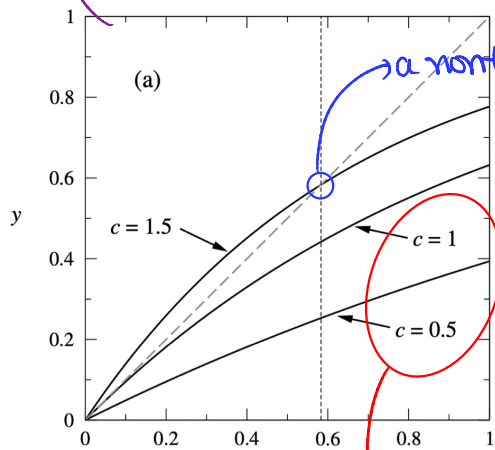
$$\therefore u = e^{-C(1-u)}$$

now, our original interest,
the fraction of the giant component $S=1-u$

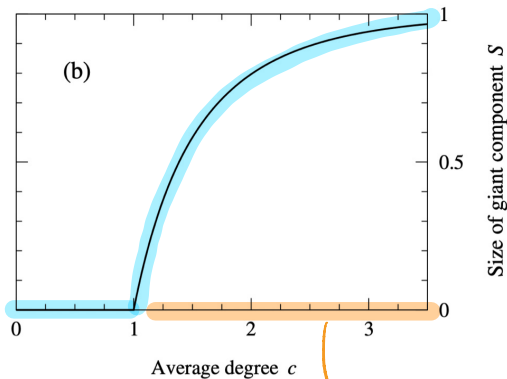
$$\Rightarrow 1-S = e^{-cS} \quad \text{or} \quad \boxed{S = 1 - e^{-cS}}$$

a self-consistency equation
for the giant component size
(first given by Erdős and Rényi
in 1959)

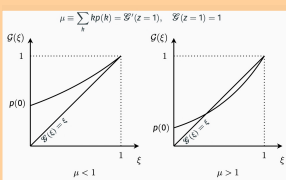
the "graphical"
solution:
 $y=S$ vs $y=1-e^{-cS}$



a non-trivial solution exists!



cf) Prof. S.-C. Park's ^Slecture



only trivial
solution $S=0$
exists! (no giant component)

what about
this?
check out
Newman's book
for this...

the "critical" point:

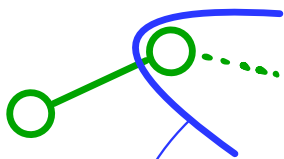
when the slope of $1 - e^{-cs}$ becomes 1 at $s=0$

$$\Rightarrow \left. \frac{d}{ds} (1 - e^{-cs}) \right|_{c^*, s=0} = c^* e^{-cs} \Big|_{s=0} = c^* = 1$$

$$\Rightarrow c^* = 1$$

the giant component starts to appear
when the average degree = 1
(a continuous phase transition)

intuitively understandable, in fact:



in fact, this part
corresponds to
the "excess degree":

(excluding the edge from the "source" node)

→ for a "walk" to be
continued, there should be
at least a single edge
for each node, on average!

→ we'll see in detail, with the
configuration model later...

the average excess degree =

the average degree in ER

recall $S = 1 - e^{-cS}$:

near $S=0$,

$$S \approx 1 - (1 - cS - \frac{1}{2} c^2 S^2)$$

$$= cS - \frac{1}{2} c^2 S^2$$

$$\Rightarrow 1 = c - \frac{1}{2} c^2 S,$$

$$S = \frac{2(c-1)}{c^2} \propto (c - c^*)^1 \propto (p - p^*)^1$$

$$p^* = \frac{1}{n-1}$$

$$S \propto (p - p_c)^\beta$$

the critical exponent $\beta = 1$

in agreement with the mean-field universality class of percolation

the "average (non-percolating) cluster size" to which a randomly chosen node belongs

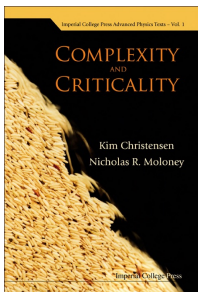


Table 1.2 The values of the critical exponents for percolation in dimensions $d = 1, 2, 3, 4, 5$, $d \geq 6$, and on the Bethe lattice. The critical exponents in $d = 3, 4, 5$ are not known exactly but the current best numerical results are listed with the uncertainty on the last digit(s) given by the figure(s) in the brackets. Two of the critical exponents have been measured numerically and the remaining critical exponents are evaluated from scaling relations.

Exponent: Quantity	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d \geq 6$	Bethe
$\beta: P_\infty(p) \propto (p - p_c)^\beta$	0 (dis)	5/36	0.4181(8)	0.657(9)	0.830(10)	1	1
$\gamma: \chi(p) \propto p - p_c ^{-\gamma}$	1	43/18	1.793(3)	1.442(16)	1.185(5) ^a	1	1
$\nu: \xi(p) \propto p - p_c ^{-1/\nu}$	1	4/3 ^b	0.8765(16)	0.689(10) ^c	0.569(5)	1/2	1/2
$\sigma: s_\xi(p) \propto p - p_c ^{-1/\sigma}$	1	36/91	0.4522(8) ^d	0.476(5)	0.496(4)	1/2	1/2
$\tau: n(s, p) \propto s^{-\tau} G(s/s_\xi)$	2	187/91	2.18906(6) ^d	2.313(3) ^e	2.412(4) ^e	5/2	5/2
$D: s_\xi \propto \xi^D$	1	91/48 ^b	2.523(6)	3.05(5)	3.54(4)	4	4

^a[Adler *et al.*, 1990].

^b[den Nijs, 1979; Nienhuis, 1982].

^c[Ballesteros *et al.*, 1997].

^d[Ballesteros *et al.*, 1999].

^e[Paul *et al.*, 2001].

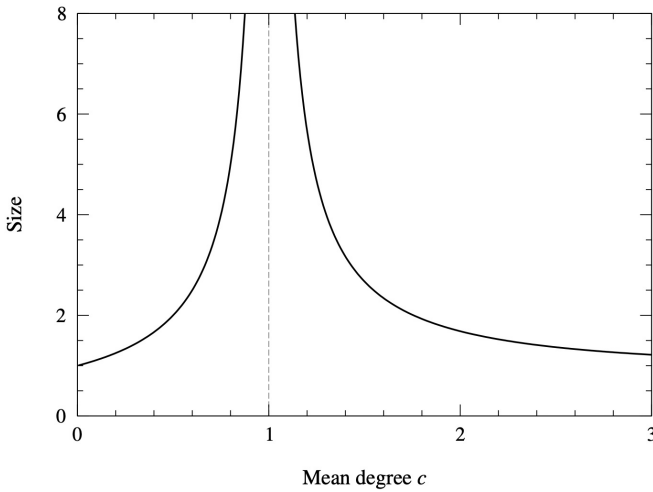


Figure 11.5: Average size of the small components in a random graph. The average size $\langle s \rangle$ of the component to which a randomly chosen node in a small component belongs, calculated from Eq. (11.29).

$$\langle s \rangle = \frac{1}{1-c+cS} \quad : \text{derivation ref) Sec. 11.6 of Newman's book (2/e)}$$

near c^* from below, $S=0$

$$\text{so } \langle s \rangle \approx \frac{1}{1-c} = (1-c)^{-1} \Rightarrow \text{the critical exponent } \gamma = 1$$

near c^* from above, $S \approx \frac{2}{c^2}(c-1)$,

$$\text{so } \langle s \rangle \approx \frac{1}{1-c+\frac{2}{c}(c-1)} = \frac{1}{c-1} \frac{1}{(-1+\frac{2}{c})} \approx (c-1)^{-1} \quad \uparrow \quad c \approx 1^+$$

from Prof. Young-Sul Cho's lecture

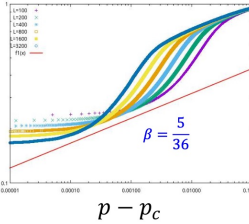
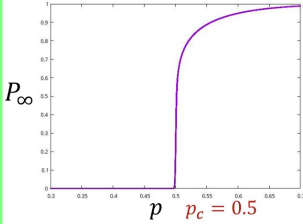


Critical phenomena of bond percolation

(2d) square lattice

$$P_{\infty}(p) \propto (p - p_c)^{\beta} \quad \text{as } p \rightarrow p_c^+$$

$$\beta = \frac{5}{36}$$



Critical phenomena of bond percolation

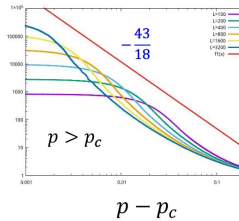
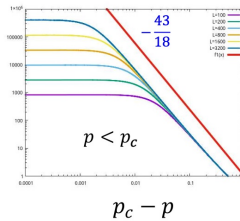
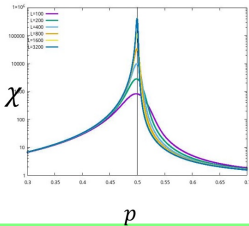
(2d) square lattice

$$\chi = \frac{\sum_i s_i^2 n_i}{\sum_i s_i n_i} \quad \text{Average cluster size}$$

$$\gamma = \frac{43}{18}$$

Average size of a finite cluster to which a randomly selected node belongs

$$\chi \propto |p - p_c|^{-\gamma} \quad \text{as } p \rightarrow p_c$$



the same universality class!

Origin of critical phenomena of bond percolation: Exact solutions

(2) Bond percolation on the Bethe lattice

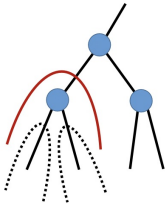
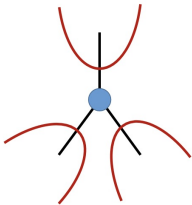
$$p_c = \frac{1}{z-1}$$

$$P_{\infty}(p) \propto (p - p_c) \quad \text{as } p \rightarrow p_c^+$$

$$\beta = 1$$

For $z = 3$

$$P_{\infty} = 1 - \left(\frac{1-p}{p}\right)^3 \quad \text{for } p \geq \frac{1}{2}$$



$$P_{\infty} = 1 - (1 - pA)^z$$

$$A = 1 - (1 - pA)^{z-1}$$

A : probability that a branch is connected to the infinite cluster

ER Bethe

$u \quad 1-A$

$n-1 \quad z-1$

$\hookrightarrow p_c = \frac{1}{n-1}$

$$u = \left[1 - \frac{C}{n-1} (1-u)\right]^{n-1}$$