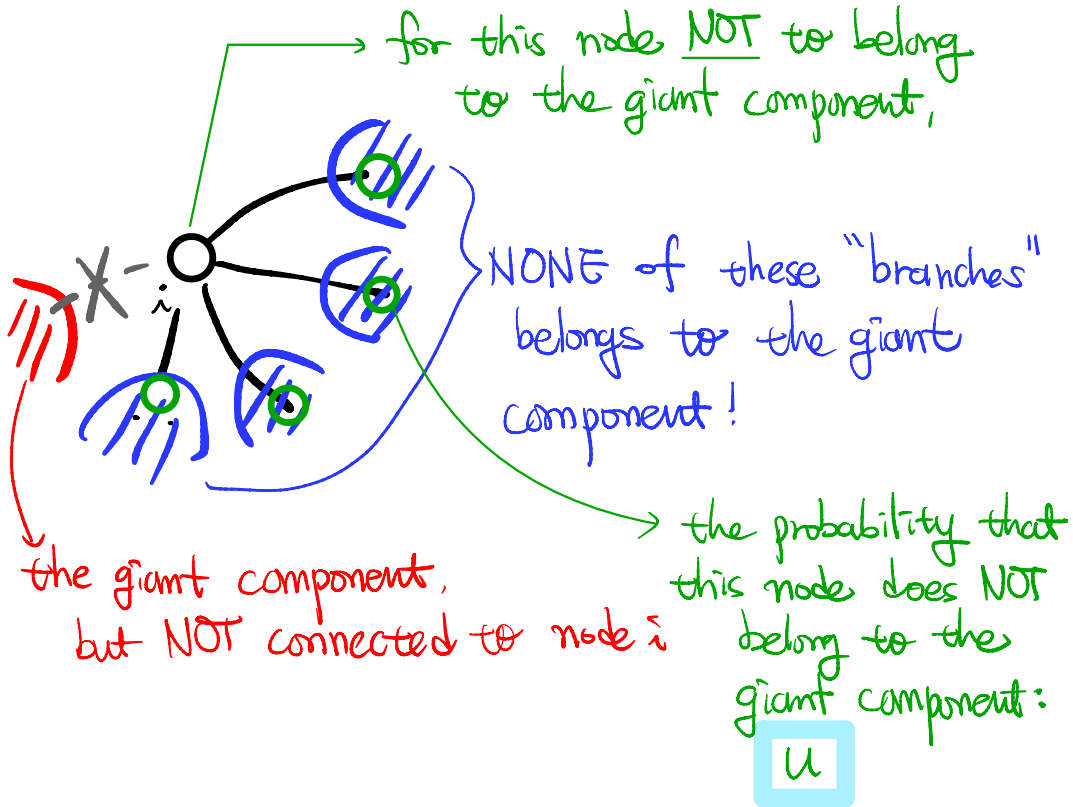
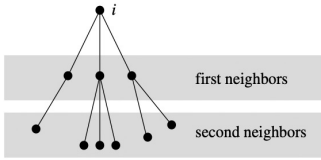


the locally tree-like configuration model w/
the degree distribution p_k



there's nothing special about the node in different branches: all of the nodes are independently realized

$\Rightarrow$ the probability for node i NOT to belong to the giant component $= u^k$ ($\because$ the degree of $i = k$)

→ on average (over all of the nodes),

the probability for a node NOT to belong to the giant component

$$= \langle u^k \rangle \quad \text{averaged over all of the nodes} \quad \xrightarrow{\sum_k p_k(\dots)} \text{in terms of degree values}$$

$$= \sum_k p_k u^k \equiv g_0(u) : \text{the probability generating function}$$

∴ $S = 1 - g_0(u)$: the fraction of the giant component

recall: for the ER random graph,

$$p_k = e^{-c} \frac{c^k}{k!} \quad (\text{the Poisson degree distribution with the average degree } c)$$

$$\text{then, } g_0(u) = \sum_k e^{-c} \frac{(cu)^k}{k!} = e^{-c} e^{cu} = e^{c(u-1)}$$

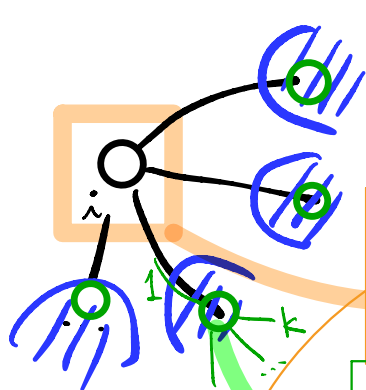
$$\Rightarrow S = 1 - e^{c(u-1)}, \text{ or } S = 1 - e^{-cS}$$

for $S = 1 - u$ in the ER random graph!

↑ $u = g_0(u)$ in ER, as we'll see soon...

However, in this more general case,

$$u \neq g_0(u) !!!$$



why?

because

the degree distribution of
node i and

the degree distribution of
node i's neighbors are
different!

P_k : the probability that a randomly
chosen node has the degree k

g_k : the probability that a randomly
chosen node's random neighbor has
the degree $(k+1)$

$$\Rightarrow g_k = \frac{(k+1) P_{k+1}}{\langle k \rangle}$$

↳ the "default" edge from
the original node

$\Rightarrow U = \sum_k g_k u^k \equiv g_1(u)$: the probability generating function for the excess degree distribution

a mathematical identity:

$$g_1(u) = \sum_{k=0}^{\infty} g_k u^k = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} (k+1) P_{k+1} u^k$$

$$= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P_k u^{k-1} = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} k P_k u^{k-1}$$

$$\begin{array}{l} \uparrow \\ k+1 \rightarrow k \\ \text{(dummy index)} \end{array} = \frac{1}{\langle k \rangle} \frac{d}{du} \underbrace{\sum_{k=0}^{\infty} P_k u^k}_{g_0(u)} = \frac{1}{\langle k \rangle} g'_0(u)$$

$$\text{and } \langle k \rangle = \sum_{k=0}^{\infty} k P_k = g'_0(u=1) = g'_0(1)$$

$$(\text{or, } \frac{g_1(1)}{\langle k \rangle} = \frac{1}{\langle k \rangle} g'_0(1) = 1)$$

↳ the sum of probabilities = 1 by normalization

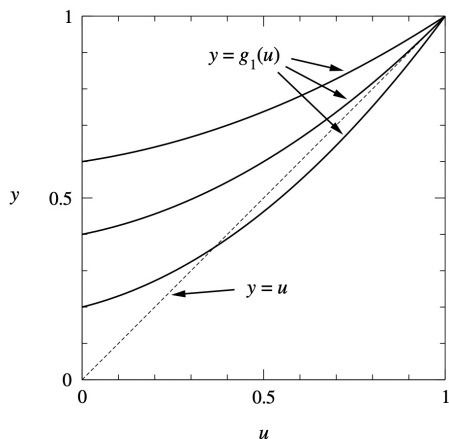
$$\therefore g_1(u) = \frac{g'_0(u)}{g'_0(1)}$$

⇒ the recipe for S :

from p_k , calculate $g_0(u) = \sum_k p_k u^k$

from $g_0(u)$, calculate $g_1(u) = g'_0(u) / g'_0(1)$

from $g_1(u)$, solve the self-consistency equation $u = g_1(u)$



⇒ from u ,

obtain $S = 1 - g_0(u)$

the ER random graph revisited:

$$g_0(u) = e^{c(u-1)} \quad (\because P_k = e^{-c} \frac{c^k}{k!})$$

$$\Rightarrow g'_0(u) = c e^{c(u-1)}, \quad g'_0(1) = c$$

$$\therefore g_1(u) = \frac{1}{c} c e^{c(u-1)} = e^{c(u-1)} = g_0(u)$$

and $u = g_1(u) = g_0(u)$, as mentioned before...

$$\begin{aligned} &\left\{ \begin{array}{l} s = 1 - u \\ s = 1 - e^{-cs} \end{array} \right. \& u = e^{c(u-1)} \\ &S = 1 - e^{-cS} \end{aligned}$$

meaning:

the degree distribution P_k
= the excess degree distribution

$$\begin{aligned} q_k &= \frac{1}{\langle k \rangle} (k+1) P_{k+1} = \frac{1}{c} (k+1) e^{-c} \frac{c^{k+1}}{(k+1)!} \\ &= e^{-c} \frac{c^k}{k!} = P_k \end{aligned}$$

For instance, the degree distribution of Erdős-Rényi (ER) graphs [11,12] is Poissonian. Therefore,

$$\begin{aligned} G_0(1-S') &= \sum_k \frac{e^{-z} z^k}{k!} (1-S')^k = e^{-zS'} \\ &= G_1(1-S'). \end{aligned} \quad (4)$$

Thus, $S = S'$ and eqs. (2) and (3) give simply $S = 1 - e^{-cS}$. Defining $f(S) = S - 1 + e^{-cS}$, one can find the solution $S(z)$ by solving $f(S) = 0$ graphically, as shown in fig. 1. A continuous ("second-order") phase transition is clearly evident in the inset of fig. 1.

S.-W. Son *et al.*,
EPL 97, 16006 (2021).



→ for a "walk" to be continued, there should be at least a single edge for each node, on average!

→ we'll see in detail, with the configuration model later...
the average excess degree =
the average degree in ER

In fact, this part corresponds to the "excess degree": (excluding the edge from the "source" node)

→ from the ER percolation note...

the power-law degree distribution

$$P_k = \begin{cases} 0 & \text{for } k=0 \\ k^{-\gamma} / \zeta(\gamma) & \text{for } k \geq 1 \end{cases}$$

) → the "pure" power-law

↪ $\zeta(\gamma) = \sum_{k=1}^{\infty} k^{-\gamma}$: the Riemann zeta function for normalization

$$\Rightarrow g_0(u) = \frac{1}{\zeta(\gamma)} \sum_{k=1}^{\infty} k^{-\gamma} u^k \rightarrow \begin{cases} g_0'(u) = \frac{1}{\zeta(\gamma)} \sum_{k=1}^{\infty} k^{-\gamma+1} u^{k-1} \\ g_0'(1) = \frac{\zeta(\gamma-1)}{\zeta(\gamma)} = \langle k \rangle \end{cases}$$

$$\rightarrow g_1(u) = \frac{g_0'(u)}{g_0'(1)} = \frac{1}{\zeta(\gamma-1)} \sum_{k=1}^{\infty} k^{-\gamma+1} u^{k-1}$$

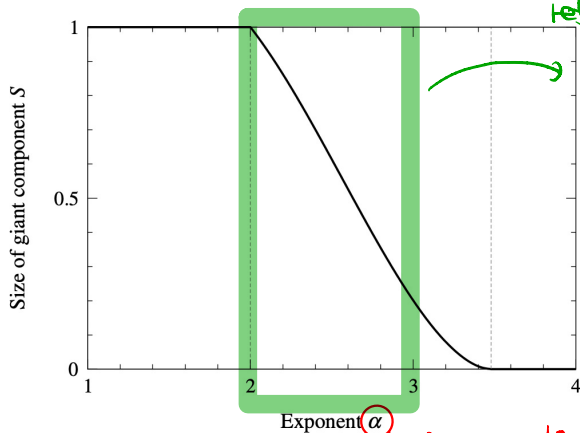
So, the self-consistency equation for u :

$$u = \frac{1}{\zeta(\gamma-1)} \sum_{k=1}^{\infty} k^{-\gamma+1} u^k$$

↪ no closed form solution, but

for $\gamma \leq 2$, $\zeta(\gamma-1) \rightarrow \infty$, so $u=0$ is the only solution

$\Rightarrow S = 1 - g_0(0) = 1$: the giant component fills the entire network!



ref) Sec. 12.5 of Newman's
for $2 < \gamma \leq 3$, book
the actual condition
for the existence
of the giant component
 $\langle k^2 \rangle - 2\langle k \rangle > 0$ is
always satisfied!
($\because \langle k^2 \rangle \rightarrow \infty$ & $\langle k \rangle$ finite)
 γ in this note (the degree
exponent)

Figure 12.6: Size of the giant component for the configuration model with a power-law degree distribution. This plot shows the fraction of the network filled by the giant component as a function of the exponent α of the power law, calculated by numerical solution of Eqs. (12.27) and (12.65). The dotted lines mark the value $\alpha = 2$, below which the giant component has size 1, and the value $\alpha = 3.4788$, above which there is no giant component.

the average excess
degree condition:
 $\sum_{k=0}^{\infty} k g(k) = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} k(k+1) P_{k+1}$
 $= \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} (k-1) k P_k = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$

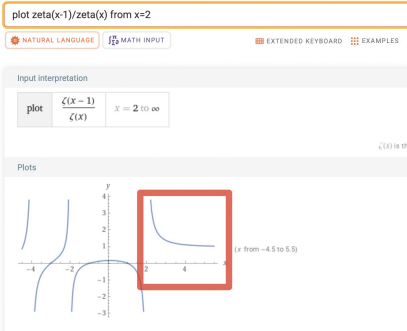
note: because we set $k_{\min} = 1$,
the average degree is changed over different γ
values!

(not quite a "fair" comparison above)

$$\langle k \rangle = \frac{S(\gamma-1)}{S(\gamma)}$$

FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA

WolframAlpha



$$g'_i(u) = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} k(k+1) P_{k+1} u^{k-1}$$

$$\leadsto g'_i(1) = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}$$

$\Rightarrow$ the average branching